

A Twisted Non-compact Elliptic Genus

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Abstract

We give a detailed path integral derivation of the elliptic genus of a supersymmetric coset conformal field theory, further twisted by a global $U(1)$ symmetry. It gives rise to a Jacobi form in three variables, which is the modular completion of a mock modular form. The derivation provides a physical interpretation to the non-holomorphic part as arising from a difference in spectral densities for the continuous part of the right-moving bosonic and fermionic spectrum. The spectral asymmetry can also be read off directly from the reflection amplitudes of the theory. By performing an orbifold, we show how our twisted elliptic genus generalizes an existing example.

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1 Introduction

The elliptic genus of two-dimensional conformal field theories with two supersymmetries provides useful information about their spectrum [1][2]. It is protected by supersymmetry, and can be computed in a convenient corner of moduli space. To acquire more information, the elliptic genus can be further twisted by global symmetries of the model. Elliptic genera have applications to anomaly calculations in string theory, to algebraic geometry, to the study of renormalization group flows, as well as to black hole entropy counting (see e.g. [3][4][5][6][7]).

Elliptic genera of sigma-models with non-compact target spaces exhibit further subtleties compared to their compact counterparts. They give rise to Jacobi forms which contain non-holomorphic contributions necessary to render mock modular holomorphic expressions modular [8][9]. In physical applications, these contributions are often postulated under the hypothesis that a certain duality group, the modular group, is a symmetry of the model.

In [10] a derivation of the non-holomorphic part of the elliptic genus of an orbifold of the non-compact coset conformal field theory $SL(2, \mathbb{R})/U(1)$ was given. The path integral result was modular, and contained both the mock modular contributions as well as its non-holomorphic completion. The elliptic genus functioned as a modular covariant infrared cut-off to the bulk partition function, exhibiting localized supersymmetric contributions, as well as contributions from long multiplets. The latter were made possible through the cancellation of a volume divergence with a fermion zero-mode.

In this paper, we will derive the path integral result in more detail, and directly in the axial coset conformal field theory in section 2. We further twist the elliptic genus with a global $U(1)$ symmetry of the model. In section 3 we lay the link to the theory of mock modular forms. We moreover provide in section 4 an independent derivation of the measure of integration in the non-holomorphic remainder term via the evaluation of the spectral asymmetry as the derivative of the difference in phase shifts for right-moving bosons and fermions. We also recuparate and generalize the results of [10] by performing a \mathbb{Z}_k orbifold on the axial coset, where \mathbb{Z}_k is a subgroup of the $U(1)_R$ symmetry of the model.

In recent independent work [11], the results of [10] were also extended towards the axial coset, and further to models with fractional levels as well as to orbifold sectors. One main point of [11] is an analysis of the ordinary bulk partition function and the modular completion of the discrete contribution to the partition function. We concentrate on the extension of the results of [10] to include a new global $U(1)$ twist.

2 The path integral

In this section, we compute the path integral of the supersymmetric coset conformal field theory $SL(2, \mathbb{R})/U(1)$ at integer level k , twisted by left R-charge as well as a global $U(1)$ symmetry. This is heavily based on the calculation of the bosonic bulk partition function [12] as well as the treatment of the supersymmetric model in [13], though there are differences in the details. See [14] for a construction of the supersymmetric partition function by the technique of deformation. A generalization of our analysis at least for fractional levels k exists.

2.1 The Supersymmetric Axially Gauged WZW model

2.1.1 Bosons and Fermions

The action S_b for the bosonic axially gauged WZW model can be written as $S_b = \kappa I(g, A)$ where we define [15]:

$$I(g, A) = I(g) + \frac{1}{2\pi} \int d^2z \operatorname{Tr} \left[2A_z \bar{\partial} g g^{-1} - 2\tilde{A}_{\bar{z}} g^{-1} \partial g + 2g^{-1} A_z g \tilde{A}_{\bar{z}} - A_z A_{\bar{z}} - \tilde{A}_{\bar{z}} \tilde{A}_z \right], \quad (2.1)$$

and κ is a constant prefactor. Here $I(g)$ refers to the action of the WZW model with the group valued map g :

$$I(g) = \frac{1}{2\pi} \int d^2z \operatorname{Tr} (g^{-1} \partial g g^{-1} \bar{\partial} g) + \frac{i}{12\pi} \int d^3z \operatorname{Tr} ((g^{-1} dg)^3). \quad (2.2)$$

The gauge fields are defined such that

$$A_z = T_a A_z^a \quad A_{\bar{z}} = T^a A_{\bar{z}}^a \quad \tilde{A}_z = \tilde{T}_a A_z^a \quad \tilde{A}_{\bar{z}} = \tilde{T}_a A_{\bar{z}}^a. \quad (2.3)$$

The notations T^a and \tilde{T}^a refer to the generators of the Lie algebra whose exponentiation is the gauged subgroup $H \subset G$. There are only two independent gauge field components: the gauge field \tilde{A} differs only in the distinct embedding of the generators \tilde{T}^a into the gauged subgroup. We introduce the subgroup-valued fields $h, \tilde{h}, \bar{h}, \tilde{\bar{h}}$ such that:

$$\begin{aligned} A_z &= \partial h h^{-1} \quad \text{and} \quad A_{\bar{z}} = \bar{\partial} \bar{h} \bar{h}^{-1} \\ \tilde{A}_z &= \partial \tilde{h} \tilde{h}^{-1} \quad \text{and} \quad \tilde{A}_{\bar{z}} = \bar{\partial} \tilde{\bar{h}} \tilde{\bar{h}}^{-1}, \end{aligned} \quad (2.4)$$

and obtain the action in the form

$$\begin{aligned} I(g, A) - I(g) &= \frac{1}{2\pi} \int d^2z \operatorname{tr} \left[2\partial h h^{-1} \bar{\partial} g g^{-1} - 2\bar{\partial} \tilde{\bar{h}} \tilde{\bar{h}}^{-1} g^{-1} \partial g \right. \\ &\quad \left. + 2\partial h h^{-1} g \bar{\partial} \tilde{\bar{h}} \tilde{\bar{h}}^{-1} g^{-1} - \bar{\partial} \bar{h} \bar{h}^{-1} \partial h h^{-1} - \bar{\partial} \tilde{\bar{h}} \tilde{\bar{h}}^{-1} \partial \tilde{h} \tilde{h}^{-1} \right]. \end{aligned} \quad (2.5)$$

The action has the following gauge symmetry:

$$\begin{aligned} g &\rightarrow m g \tilde{m}^{-1} \\ h &\rightarrow m h \quad \bar{h} \rightarrow m \bar{h} \\ \tilde{h} &\rightarrow \tilde{m} \tilde{h} \quad \tilde{\bar{h}} \rightarrow \tilde{m} \tilde{\bar{h}}, \end{aligned} \quad (2.6)$$

where m takes values in the gauge group H and \tilde{m} is defined in terms of m as above. We consider the gauged subgroup H to be $U(1)$ and pick the anomaly free axial gauging $\tilde{T} = -T$. Substituting this into the expression for the gauge field we find that we have the relations:

$$\tilde{h} = h^{-1} \quad \text{and} \quad \tilde{\bar{h}} = \bar{h}^{-1}. \quad (2.7)$$

We will study the theory in which the group element g after analytic continuation takes values in Euclidean AdS_3 . This is the hyperbolic three-plane corresponding to the space of 2×2 hermitian matrices g with unit determinant. We will use the following parameterization of these matrices:

$$g = \begin{pmatrix} e^{-\phi} & v \\ \bar{v} & e^{\phi}(1 + v\bar{v}) \end{pmatrix}. \quad (2.8)$$

Substituting this into the WZW action, one can write down the world sheet action for the bosonic fields:

$$I(g) = \frac{\kappa}{\pi} \int d^2 z \left(\partial \phi \bar{\partial} \phi + (\partial \bar{v} + \bar{v} \partial \phi)(\bar{\partial} v + v \bar{\partial} \phi) \right). \quad (2.9)$$

The axially gauged action, with gauge field A given as in equation (2.3) takes the form

$$S_b(\phi, v, \bar{v}, A) = \frac{\kappa}{\pi} \int d^2 z (\partial \phi + A_z)(\bar{\partial} \phi + A_{\bar{z}}) + (\partial_z + \partial_z \phi + A_z) \bar{v} (\bar{\partial} + \bar{\partial} \phi + A_{\bar{z}}) v. \quad (2.10)$$

A supersymmetric extension of the axially gauged action by worldsheet fermions is provided by the addition of the fermionic action:

$$S_f(\psi_{\pm}, A) = \frac{\kappa}{\pi} \int d^2 z [\psi^- (\partial_{\bar{z}} + A_{\bar{z}}) \psi^+ + \tilde{\psi}^- (\partial_z + A_z) \tilde{\psi}^+]. \quad (2.11)$$

The supersymmetry of the axial coset theory was analyzed in detail in [16].

2.1.2 The $U(1)_R$ and a global $U(1)$ symmetry

The axial coset model has $(2, 2)$ supersymmetry. In order to calculate the elliptic genus we must identify the global $U(1)_R$ symmetry which is part of the left-moving $N = 2$ algebra. We follow the analysis in [17] in order to calculate the $U(1)_R$ charges of the fields. Let us consider a $U(1)$ transformation with parameter γ that acts on the fermions as follows:

$$\begin{aligned} \psi^+ &\rightarrow e^{i\gamma c} \psi^+ & \psi^- &\rightarrow e^{-i\gamma c} \psi^- \\ \tilde{\psi}^+ &\rightarrow e^{-i\gamma \tilde{c}} \tilde{\psi}^+ & \tilde{\psi}^- &\rightarrow e^{i\gamma \tilde{c}} \tilde{\psi}^-, \end{aligned} \quad (2.12)$$

where c and \tilde{c} are arbitrary parameters for now. We also postulate the following transformation for the bosonic fields in the axially gauged WZW model:

$$\delta g = i\gamma(\tilde{x}Tg + xgT) \quad \text{and} \quad \delta A_{z, \bar{z}} = 0. \quad (2.13)$$

Here T is the Lie algebra valued matrix that is along the gauged direction. There are three conditions imposed on the four variables $\{\tilde{c}, c, \tilde{x}, x\}$. The first follows from imposing that the action be invariant under the above transformations. The axially gauged bosonic action is classically non-invariant. Let us consider its variation, setting $x = 0$ for now:

$$\delta g = i\gamma \tilde{x} T g \quad \delta g^{-1} = -i\gamma \tilde{x} g^{-1} T \quad (2.14)$$

The gauge field dependent terms vary as follows:

$$\begin{aligned} \frac{\pi}{\kappa} \delta \mathcal{L}_B = \text{Tr} \left(A_{\bar{z}} [-i\gamma \tilde{x} g^{-1} T \partial_z g + g^{-1} \partial_z (i\gamma \tilde{x} T g)] - A_z [\partial_{\bar{z}} (i\gamma \tilde{x} T g) g^{-1} - \partial_{\bar{z}} g (i\gamma \tilde{x} g^{-1} T)] \right) \\ + \text{Tr} [A_{\bar{z}} (-i\gamma \tilde{x} g^{-1} T) A_z g + A_{\bar{z}} g^{-1} A_z (i\gamma \tilde{x} T g)]. \end{aligned} \quad (2.15)$$

Both the generator T and the gauge fields $A_{z, \bar{z}}$ are proportional to the same Lie algebra element, so they commute with each other. Using this, one can see that the two terms in the second line cancel each other out. Let us consider the term proportional to the gauge field component A_z in the first line. Using integration by parts, we obtain for this term:

$$\text{Tr} [\partial_{\bar{z}} A_z (i\gamma \tilde{x} T) + A_z (i\gamma \tilde{x} T g) \partial_{\bar{z}} g^{-1} + A_z (i\gamma \tilde{x} \partial_{\bar{z}} g) g^{-1} T]. \quad (2.16)$$

Once again, commuting the generator T through the gauge field component A_z , the last two terms cancel. A similar calculation can be done for the terms proportional to the gauge field component $A_{\bar{z}}$. Putting them together, we find that

$$\delta \mathcal{L}_B = (-i\gamma \tilde{x}) \frac{\kappa}{\pi} \text{Tr} (F_{z\bar{z}} T). \quad (2.17)$$

A similar calculation can be done for the variation proportional to x , leading to

$$\delta S_B = \kappa \delta I_A(g, A) = i\gamma(x - \tilde{x}) \frac{\kappa}{\pi} \int d^2 z F_{z\bar{z}}. \quad (2.18)$$

The fermionic part is invariant classically but it has a quantum anomaly due to the chiral anomaly in two dimensions. The anomalous variation is:

$$\delta S_F = 2i\gamma(\tilde{c} + c) \frac{1}{\pi} \int d^2 z F_{z\bar{z}}. \quad (2.19)$$

Invariance of the action then imposes the constraint:

$$\kappa(\tilde{x} - x) = 2(\tilde{c} + c). \quad (2.20)$$

The remaining two constraints follow when we identify the $\mathcal{N} = 2$ supersymmetries in the axially gauged model and impose that the R-charge transformation commute with the right moving supersymmetries and has unit charge under the left-moving supersymmetries. This leads to the conditions:

$$x = -c \quad \text{and} \quad \tilde{x} = \tilde{c} - 1. \quad (2.21)$$

The gauge symmetry of the action can be used to impose the constraint $x = -\tilde{x}$. Putting all this together, we find the solution

$$c = \frac{1}{\kappa - 2} = \frac{1}{k} \quad \text{and} \quad \tilde{c} = \frac{\kappa - 1}{\kappa - 2} = \frac{k + 1}{k}, \quad (2.22)$$

where $k = \kappa - 2$ is the supersymmetric level of the coset. These equations determine the R-charges of the fermions, while the charges of the bosons are given by

$$x = -\frac{1}{k} \quad \text{and} \quad \tilde{x} = \frac{1}{k}. \quad (2.23)$$

This finalizes the determination of the left $U(1)_R$ action.

In addition, we have a global $U(1)$ symmetry that acts as a rotation on the fermions and on the complex field v :

$$\psi^\pm \rightarrow e^{\mp i\lambda} \psi^\pm \quad \tilde{\psi}^\pm \rightarrow e^{\mp i\lambda} \tilde{\psi}^\pm \quad v \rightarrow e^{-i\lambda} v. \quad (2.24)$$

This is an isometric $U(1)$ rotation of the coset tangent space in which these fields (or their derivatives) take values.

Our first goal is to compute the elliptic genus χ_{cos} , twisted by the above global $U(1)$ symmetry:

$$\chi_{cos}(q, z, y) = \text{Tr} \left((-1)^F q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} z^{J_0^R} y^Q \right), \quad (2.25)$$

where by J_0^R and Q , we denote the zero-mode of the left R-current and the global $U(1)$ charge respectively. We will also use the notations $z = e^{2\pi i\alpha}$ and $y = e^{2\pi i\beta}$ for the chemical potentials. We will refer to the above quantity as the twisted elliptic genus. We compute it in a Lagrangian picture using the path integral formalism. This involves computing the partition function of the worldsheet theory on a torus. The effect of the charge insertions in the twisted elliptic genus will be to change the periodicity conditions of the charged fields in the path integral. Modular covariance will be manifest at all stages.

2.1.3 Parameterizing the gauge field with holonomies

We will study the worldsheet theory on a torus, with the worldsheet coordinates identified under the operations $(z, \bar{z}) \sim (z + 2\pi, \bar{z} + 2\pi) \sim (z + 2\pi\tau, \bar{z} + 2\pi\bar{\tau})$. The parameter $\tau = \tau_1 + i\tau_2$ is the modular parameter of the torus. To parameterize the group elements h and \bar{h} in terms of which we defined the gauge field, we introduce the function Φ :

$$\begin{aligned} \Phi(z, \bar{z}) &= \frac{i}{2\tau_2} [s_1(z\bar{\tau} - \bar{z}\tau) - s_2(\bar{z} - z)] \\ &= \frac{i}{2\tau_2} (z\bar{u} - \bar{z}u), \end{aligned} \quad (2.26)$$

where we have defined $u = s_1\tau + s_2$. We take the holonomies $s_{1,2}$ to satisfy $0 \leq s_1, s_2 < 1$. The function $\Phi(z, \bar{z})$ is a real harmonic function with the following periodicity:

$$\Phi(z + 2\pi, \bar{z} + 2\pi) = \Phi(z, \bar{z}) + 2\pi s_1 \quad \text{and} \quad \Phi(z + 2\pi\tau, \bar{z} + 2\pi\bar{\tau}) = \Phi(z, \bar{z}) - 2\pi s_2. \quad (2.27)$$

There is an inherent ambiguity in the definition of the gauge field defined as in equation (2.4). The function h can be multiplied by a purely anti-holomorphic function and it does not affect the expression for the gauge field A_z and similarly, the field \bar{h} can be multiplied by a holomorphic function without changing the gauge field $A_{\bar{z}}$. We make a particular choice to fix this ambiguity and parameterize the group elements h and \bar{h} which lead to the gauge field in equation (2.4) as follows:

$$\begin{aligned} h(z, \bar{z}) &= e^{(X-iY)T} h^u \\ \bar{h}(z, \bar{z}) &= (h(z, \bar{z}))^\dagger = (h^u)^\dagger e^{(X+iY)T}. \end{aligned} \quad (2.28)$$

where we have defined

$$h^u = e^{\frac{-T}{2\tau_2}\bar{u}(z-\bar{z})} \quad \text{and} \quad (h^u)^\dagger = e^{\frac{T}{2\tau_2}u(z-\bar{z})}. \quad (2.29)$$

The generator T is the generator of the $U(1)$ subgroup that is being gauged. The scalar field X corresponds to a non-compact direction while Y is a compact boson which has non-trivial windings around the cycles of the torus. The group elements \tilde{h} and $\tilde{\bar{h}}$ are obtained from equation (2.28) by a sign flip of the generator T . With these definitions, the gauge fields take the form

$$\begin{aligned} A_z &= \partial X - i\partial Y - \frac{\bar{u}}{2\tau_2} \\ &= \partial X - i\partial Y^u. \end{aligned} \quad (2.30)$$

Here we have defined

$$Y^u(z, \bar{z}) = Y(z, \bar{z}) + \Phi(z, \bar{z}). \quad (2.31)$$

Similarly, the anti-holomorphic component of the gauge field becomes

$$A_{\bar{z}} = \bar{\partial} X + i\bar{\partial} Y - \frac{u}{2\tau_2} = \bar{\partial} X + i\bar{\partial} \bar{Y}^u. \quad (2.32)$$

2.2 The Twisted Partition Function

We compute the twisted elliptic genus in the path integral formalism. We will discuss in detail the precise periodicity conditions to be imposed in the next subsection. We denote the partition function as

$$\chi_{cos}(q, z, y) = \int_{\Sigma} d^2u \int [\mathcal{D}g] \int [\mathcal{D}X\mathcal{D}Y] \int [\mathcal{D}\psi_{\pm}\mathcal{D}\tilde{\psi}^{\pm}] e^{-\kappa I_A(g,A)} e^{-S_f(\psi^{\pm}, \tilde{\psi}^{\pm}, A)} \Big|_{TPC}, \quad (2.33)$$

where the subscript refers to the twisted periodicity conditions. The fermionic measure is defined so as to respect the axial gauging. The integral over the gauge field has been broken up into an ordinary integral over the holonomy u and the integral over the scalar fields X and Y . In what follows we will show that the path integral factorizes into Gaussian integrals.

2.2.1 Breaking up the bosonic action

We start with the bosonic piece of the axially gauged model. Using the Polyakov-Wiegmann identity, we can rewrite it as follows:

$$I_A(g, h, \bar{h}) = I(h^{-1}g\tilde{\bar{h}}) - I(h^{-1}\bar{h}). \quad (2.34)$$

Substituting equation (2.28) into the action, we get

$$I_A(g, A) = I((h^u)^{-1}e^{(-X-iY)T}g e^{(-X+iY)T}((h^u)^\dagger)^{-1}) - I((h^u)^{-1}e^{-2iY T}(h^u)^\dagger). \quad (2.35)$$

Let us perform a similarity transformation on g , with

$$g \longrightarrow g' = e^{(-X-iY)T}g e^{(-X+iY)T}, \quad (2.36)$$

and add and subtract the term $I(h^u \cdot ((h^u)^\dagger)^{-1})$:

$$\begin{aligned} I_A(g, A) &= \left(I((h^u)^{-1} g' ((h^u)^\dagger)^{-1}) - I(h^u \cdot ((h^u)^\dagger)^{-1}) \right) \\ &\quad - \left(I((h^u)^{-1} e^{-2iY^T} (h^u)^\dagger) - I(h^u \cdot ((h^u)^\dagger)^{-1}) \right). \end{aligned} \quad (2.37)$$

Analogously to equation (2.34) one can also write the vector gauged actions as follows:

$$I_V(g, h, \bar{h}) = I(h^{-1} g \bar{h}) - I(h^{-1} \bar{h}). \quad (2.38)$$

Equations (2.34) and (2.38) imply that:

$$I_A(g, A) = I_V(g', h^u, ((h^u)^\dagger)^{-1}) - I_A(e^{-2iY^T}, h^u, ((h^u)^\dagger)^{-1}). \quad (2.39)$$

The key point that makes this identity useful in doing the path integral is the invariance of the measure for g as a result of which one can replace $\mathcal{D}g$ with $\mathcal{D}g'$. The path integral therefore takes the form

$$\begin{aligned} \chi_{cos} &= \int_{\Sigma} d^2u \int [\mathcal{D}g'] e^{-\kappa I_V(g', h^u, ((h^u)^\dagger)^{-1})} \\ &\quad \times \int [\mathcal{D}X][\mathcal{D}Y] e^{\kappa I_A(e^{-2iY^T}, h^u, ((h^u)^\dagger)^{-1})} \times \int [\mathcal{D}\psi_{\pm}] e^{-S_f(\psi^{\pm}, \tilde{\psi}^{\pm}, A)} \Big|_{TPC}. \end{aligned} \quad (2.40)$$

The action of the abelian compact boson Y is easy to evaluate:

$$\begin{aligned} I_A(e^{-2iY^T}, h^u, ((h^u)^\dagger)^{-1}) &= -\frac{1}{\pi} \int d^2z \left| i\partial Y + \frac{\bar{u}}{2\tau_2} \right|^2 \\ &= -\frac{1}{\pi} \int d^2z |\partial Y^u|^2, \end{aligned} \quad (2.41)$$

where we used equation (2.31).

2.2.2 Gauge degrees of freedom

The axial coset model has a gauge symmetry. The non-compact X field introduced above is precisely the gauge degree of freedom associated to this gauge symmetry, and therefore, it can be gauged away without affecting the physics. Gauge fixing the path integral leads to the addition of a (b, c) ghost system via the Fadeev-Popov procedure. The $\mathcal{D}X$ integral therefore is just the volume of the gauge group. Dividing by this volume, we end up with the path integral

$$\begin{aligned} \chi_{cos} &= \int_{\Sigma} d^2u \int [\mathcal{D}g] e^{-\kappa I_V(g, h^u, ((h^u)^\dagger)^{-1})} \times \int [\mathcal{D}Y] e^{\kappa I_A(e^{-2iY^T}, h^u, ((h^u)^\dagger)^{-1})} \\ &\quad \times \int [\mathcal{D}\psi_{\pm}] e^{-S_f(\psi^{\pm}, \tilde{\psi}^{\pm}, A)} \times \int [\mathcal{D}b \mathcal{D}c \mathcal{D}\tilde{b} \mathcal{D}\tilde{c}] e^{-S_{gh}(b, c, \tilde{b}, \tilde{c})} \Big|_{TPC}. \end{aligned} \quad (2.42)$$

At this point, the Euclidean AdS_3 action is decoupled from the remaining fields. The fermions are still coupled to the Y -field but we will disentangle these two sectors. We will write down the fully factorized partition function after we discuss the twisted periodicity conditions since both the twisting and the holonomies play a role in decoupling the fermions.

2.2.3 Twisted periodicity conditions

We now turn to describe the periodicity conditions that we impose on our fields. Since we would like to put the $U(1)_R$ twist as an operator insertion, this implies that the twist is in the time direction. Since the Hamiltonian is L_0 , the definition of the trace singles out τ as the time direction and we therefore put the

twisted periodicity condition along the τ -direction. From the R-charges and global symmetry charges of the fields we get:

$$\begin{aligned} v(z + \tau, \bar{z} + \bar{\tau}) &= e^{i(\frac{\alpha}{k} - \beta)} v(z, \bar{z}) \\ \psi^\pm(z + \tau, \bar{z} + \bar{\tau}) &= e^{\pm i(\frac{\alpha}{k} - \beta)} \psi^\pm(z, \bar{z}) \\ \tilde{\psi}^\mp(z + \tau, \bar{z} + \bar{\tau}) &= e^{\pm i(\frac{\alpha(k+1)}{k} - \beta)} \tilde{\psi}^\mp(z, \bar{z}). \end{aligned} \quad (2.43)$$

Since it is more straightforward to do the path integral over periodic fields, we redefine the bosonic field v such that it becomes periodic. We define a new periodic field v_p :

$$v_p(z, \bar{z}) = v(z, \bar{z}) e^{-\frac{\alpha}{k}(\beta)(z - \bar{z})/2\tau_2}. \quad (2.44)$$

The effect of this on the action is clear. It modifies the holonomy coupled to the field v additively. There will be a similar effect on the fermions but it is more subtle since the α -dependent periodicity conditions can only be removed by an anomalous rotation of the fermions. We describe this in detail next.

2.2.4 The Fermionic Action

The fermionic action is of the form

$$S_f(\psi_\pm, A) = \frac{\kappa}{\pi} \int d^2 z [\psi^-(\partial_{\bar{z}} + A_{\bar{z}})\psi^+ + \tilde{\psi}^-(\partial_z + A_z)\tilde{\psi}^+]. \quad (2.45)$$

Let us first perform a chiral rotation on the fermions and define new fields

$$\begin{aligned} \eta^\pm &= e^{\pm iY \pm \frac{u(z - \bar{z})}{2\tau_2}} \psi^\pm \\ \tilde{\eta}^\mp &= e^{\pm iY \pm \frac{\bar{u}(z - \bar{z})}{2\tau_2}} \tilde{\psi}^\mp \end{aligned} \quad (2.46)$$

The action now takes the form

$$S_f = \frac{\kappa}{\pi} \int d^2 z (\eta^+ \partial_{\bar{z}} \eta^- + \tilde{\eta}^+ \partial_z \tilde{\eta}). \quad (2.47)$$

The new fermions η and $\tilde{\eta}$ satisfy the periodicity conditions

$$\begin{aligned} \eta^\pm(z + 2\pi\tau, \bar{z} + 2\pi\bar{\tau}) &= e^{\pm i(u + \frac{\alpha}{k} - \beta)} \eta^\pm(z, \bar{z}) \\ \tilde{\eta}^\mp(z + 2\pi\tau, \bar{z} + 2\pi\bar{\tau}) &= e^{\pm i(\bar{u} + \frac{\alpha(k+1)}{k} - \beta)} \tilde{\eta}^\mp(z, \bar{z}) \end{aligned} \quad (2.48)$$

The chiral rotation we performed in equation (2.46) is anomalous. This means that the fermionic measure transforms as well. If the fermions were periodic to begin with, the anomaly due to the chiral rotation in equation (2.46) is given by

$$-\frac{2}{\pi} \int d^2 z |\partial Y^u|^2. \quad (2.49)$$

There is an additional contribution to the anomaly due to the twisted periodicity condition on the fermions which is a pure phase equal to:

$$\int d^2 z \frac{\alpha}{2\tau_2} (z - \bar{z}) F_{z\bar{z}}. \quad (2.50)$$

Note that the β -dependence of the boundary conditions can be removed by a non-anomalous axial rotation of the fermions. After integrating the anomalous phase by parts, it can be written as the wedge product of two one-forms:

$$-i \int \frac{\alpha}{2\tau_2} (dz - d\bar{z}) \wedge dY^u. \quad (2.51)$$

Using the Riemann bilinear identity, we find that this is equal to

$$2\pi\alpha(w + s_1). \quad (2.52)$$

Therefore the net effect of the chiral rotation that gave rise to the free fermion action is two-fold. Firstly, we get a bosonic contribution to the action, whose result is to shift the coefficient of the Y -part of the action, $\kappa \rightarrow \kappa - 2$. Secondly, we get an α -dependent phase as shown in equation (2.52). Finally, we end up with a completely factorized form of the partition function:

$$\begin{aligned}\chi_{cos} &= \int_{\Sigma} d^2u \int [\mathcal{D}g] e^{-\kappa I_V(g, h^u, ((h^u)^\dagger)^{-1})} \times \int [\mathcal{D}Y] e^{(\kappa-2)I_A(e^{-2iY^T}, h^u, ((h^u)^\dagger)^{-1})} \\ &\quad \times \int [\mathcal{D}\eta_{\pm} \mathcal{D}\tilde{\eta}^{\pm}] e^{2\pi i \alpha(w+s_1)} e^{-S_f(\eta^{\pm}, \tilde{\eta}^{\pm})} \times \int [\mathcal{D}b \mathcal{D}c \mathcal{D}\tilde{b} \mathcal{D}\tilde{c}] e^{-S_{gh}(b, c, \tilde{b}, \tilde{c})} \\ &= \int_{\Sigma} d^2u Z_g(u, \tau) Z_Y(u, \tau) Z_f(u, \tau) Z_{gh}(\tau).\end{aligned}\quad (2.53)$$

Note that the four pieces have no common factor such that the path integrals can be performed separately.

2.2.5 Evaluating the Partition Functions

The H_3^+ sector: The vector-gauged action in equation (2.40) can be obtained by relating it to an axially gauged action as follows:

$$\begin{aligned}I_V(g, h^u, ((h^u)^\dagger)^{-1}) &= I((h^u)^{-1}g((h^u)^\dagger)^{-1}) - I((h^u)^{-1}((h^u)^\dagger)^{-1}) \\ &= I_A(g, h^u, (h^u)^\dagger) + I((h^u)^{-1}(h^u)^\dagger) - I((h^u)^{-1}((h^u)^\dagger)^{-1}).\end{aligned}\quad (2.54)$$

The last two terms are easy to evaluate:

$$I((h^u)^{-1}(h^u)^\dagger) = \frac{-\pi(\text{Re } u)^2}{\tau_2} \quad \text{and} \quad I((h^u)^{-1}((h^u)^{-1})^\dagger) = \frac{\pi(\text{Im } u)^2}{\tau_2}.\quad (2.55)$$

Substituting the values, we find that

$$I_V(g, h^u, ((h^u)^\dagger)^{-1}) = I_A(g, h^u, (h^u)^\dagger) - \frac{\pi|u|^2}{\tau_2}.\quad (2.56)$$

We have already written out the action for the axially gauged coset in equation (2.10). In this case the gauge fields are purely given in terms of the holonomy. We find the action:

$$\begin{aligned}I_g &= \kappa I_V(g, h^u, ((h^u)^\dagger)^{-1}) = \frac{\kappa}{\pi} \int d^2z (\partial\phi - \frac{\bar{u}}{2\tau_2})(\bar{\partial}\phi - \frac{u}{2\tau_2}) \\ &\quad + \frac{\kappa}{\pi} \int d^2z (\partial_z + \partial_z\phi - \frac{\bar{u}}{2\tau_2})\bar{v}(\bar{\partial} + \bar{\partial}\phi - \frac{u}{2\tau_2})v - \frac{\kappa\pi|u|^2}{\tau_2}.\end{aligned}\quad (2.57)$$

We now perform the chiral rotation of the fields (v, \bar{v}) as in equation (2.44) so as to write the action in terms of periodic fields v_p and \bar{v}_p . This leads to:

$$\begin{aligned}I_g &= \frac{\kappa}{\pi} \int d^2z (\partial\phi - \frac{\bar{u}}{2\tau_2})(\bar{\partial}\phi - \frac{u}{2\tau_2}) \\ &\quad + \frac{\kappa}{\pi} \int d^2z \left(\partial + \partial\phi - \frac{\bar{u} - (\frac{\alpha}{k} - \beta)}{2\tau_2} \right) \bar{v}_p \left(\bar{\partial} + \bar{\partial}\phi - \frac{u - (\frac{\alpha}{k} - \beta)}{2\tau_2} \right) v_p - \frac{\kappa\pi|u|^2}{\tau_2}.\end{aligned}\quad (2.58)$$

The path integral for the axially gauged action has been computed previously [19]. We quote the result for the partition function:

$$Z_g(u, \tau) = \frac{\sqrt{k}\kappa}{\sqrt{\tau_2}} \frac{e^{\frac{2\pi(\text{Im } u)^2}{\tau_2}}}{|\theta_{11}(\tau, u - \frac{\alpha}{k} + \beta)|^2}.\quad (2.59)$$

The Boson Y : We recall the action of the Y -dependent piece; the only subtlety is the shift in the coefficient of the action, from $\kappa \rightarrow \kappa - 2$, which came from the anomalous rotation of the fermions:

$$S_Y = -(\kappa - 2)I_A(e^{-2iY^T}, h^u, ((h^u)^\dagger)^{-1}) = \frac{k}{\pi} \int d^2z |\partial Y^u|^2.\quad (2.60)$$

This is the action for a real compact scalar. Because of the presence of the holonomy, there is a shift in the periodicity of Y^u around the cycles of the torus:

$$\begin{aligned} Y^u(z + 2\pi, \bar{z} + 2\pi) &= Y^u(z, \bar{z}) + 2\pi(w + s_1) \\ Y^u(z + 2\pi\tau, \bar{z} + 2\pi\tau) &= Y^u(z, \bar{z}) - 2\pi(m + s_2). \end{aligned} \quad (2.61)$$

From the action, we observe that we have a twisted compact boson at radius $\sqrt{2k}$. The partition function for such a boson is given by

$$Z_Y(u, \tau) = \frac{\sqrt{k}}{\sqrt{\tau_2} |\eta(\tau)|^2} \sum_{m, n \in \mathbb{Z}} e^{-\frac{\pi k}{\tau_2} |(w + s_1)\tau + (m + s_2)|^2}. \quad (2.62)$$

The Fermions: The action for the fermionic part is of the form

$$S_f(\eta^\pm, \tilde{\eta}^\pm, a) = \frac{\kappa}{\pi} \int d^2z [\eta^+ \partial_{\bar{z}} \eta^- + \tilde{\eta}^+ \partial_z \tilde{\eta}^-]. \quad (2.63)$$

The fermions η and $\tilde{\eta}$ satisfy the periodicity conditions specified in equations (2.48). The path integral for such chiral fermions has been discussed for instance in [20] and is given by²:

$$Z_f(u, \tau) = \frac{1}{\kappa} \left[e^{-i2\pi s_1(s_2 + \frac{\alpha(k+1)}{k} - \beta)} e^{-2\pi \frac{(\text{Im } u)^2}{2\tau_2}} \frac{\theta_{11}(\tau, u - \frac{\alpha(k+1)}{k} + \beta)}{\eta(\tau)} \right] \times \quad (2.64)$$

$$\begin{aligned} & \left[e^{i2\pi s_1(s_2 + \frac{\alpha}{k} - \beta)} e^{-2\pi \frac{(\text{Im } u)^2}{2\tau_2}} \frac{\theta_{11}(\bar{\tau}, u - \frac{\alpha}{k} + \beta)}{\eta(\bar{\tau})} \right] \\ &= \frac{1}{\kappa} e^{-i2\pi s_1 \alpha} e^{-2\pi \frac{(\text{Im } u)^2}{\tau_2}} \frac{\theta_{11}(\tau, u - \frac{\alpha(k+1)}{k} + \beta) \theta_{11}(\bar{\tau}, u - \frac{\alpha}{k} + \beta)}{|\eta(\tau)|^2}. \end{aligned} \quad (2.65)$$

The Ghosts: The ghost path integral is standard:

$$Z_{gh}(\tau) = \int [D\tilde{b} D\tilde{c} D\tilde{b} D\tilde{c}] e^{-S_{gh}(b, c, \tilde{b}, \tilde{c})} = \tau_2 |\eta(\tau)|^4. \quad (2.66)$$

Putting all this together, we find that the full partition function is:

$$\chi_{cos}(\tau, \alpha, \beta) = k \int_0^1 ds_{1,2} \sum_{m, w \in \mathbb{Z}} \frac{\theta_{11}(s_1\tau + s_2 - \alpha \frac{k+1}{k} + \beta, \tau)}{\theta_{11}(s_1\tau + s_2 - \frac{\alpha}{k} + \beta, \tau)} e^{2\pi i \alpha w} e^{-\frac{k\pi}{\tau_2} |(m + s_2) + (w + s_1)\tau|^2}. \quad (2.67)$$

3 The long and short of it

There are short multiplet or discrete character contributions to the elliptic genus [13], as well as long multiplet or continuous character contributions [10]. In this section, we identify these two types of contributions to the path integral. The result of the axial coset path integral calculation was:

$$\chi_{cos} = k \int_0^1 ds_{1,2} \sum_{m, w \in \mathbb{Z}} \frac{\theta_{11}(s_1\tau + s_2 - \alpha \frac{k+1}{k} + \beta, \tau)}{\theta_{11}(s_1\tau + s_2 - \frac{\alpha}{k} + \beta, \tau)} e^{2\pi i \alpha w} e^{-\frac{k\pi}{\tau_2} |(m + s_2) + (w + s_1)\tau|^2}. \quad (3.1)$$

Recall that we have the notations $z = e^{2\pi i \alpha}$ as well as $y = e^{2\pi i \beta}$. To analyze the modular properties of the path integral result, it is convenient to work with the expression after double Poisson resummation:

$$\chi_{cos} = \int_0^1 ds_{1,2} \sum_{m, w \in \mathbb{Z}} \frac{\theta_{11}(s_1\tau + s_2 - \alpha \frac{k+1}{k} + \beta, \tau)}{\theta_{11}(s_1\tau + s_2 - \frac{\alpha}{k} + \beta, \tau)} e^{-2\pi i s_2 w} e^{2\pi i s_1(m - \alpha)} e^{-\frac{\pi}{k\tau_2} |m - \alpha + w\tau|^2}. \quad (3.2)$$

²We have chosen the phase factor for the chiral determinant such that the result is covariant under modular S-transformations.

The modular and elliptic properties can be computed as in [10]. We summarize the result:

$$\begin{aligned}
\chi_{cos}(\tau+1, \alpha, \beta) &= \chi_{cos}(\tau, \alpha, \beta) \\
\chi_{cos}\left(-\frac{1}{\tau}, \frac{\alpha}{\tau}, \frac{\beta}{\tau}\right) &= e^{\pi i \frac{\alpha}{3} \alpha^2 / \tau - 2\pi i \alpha \beta / \tau} \chi_{cos}(\tau, \alpha, \beta) \\
\chi_{cos}(\tau, \alpha+k, \beta) &= (-1)^{\frac{\alpha}{3}k} \chi_{cos}(\tau, \alpha, \beta) \\
\chi_{cos}(\tau, \alpha+k\tau, \beta) &= (-1)^{\frac{\alpha}{3}k} e^{-\pi i \frac{\alpha}{3}(k^2\tau+2k\alpha)} e^{2\pi i \beta k} \chi_{cos}(\tau, \alpha, \beta) \\
\chi_{cos}(\tau, \alpha, \beta+1) &= \chi_{cos}(\tau, \alpha, \beta) \\
\chi_{cos}(\tau, \alpha, \beta+\tau) &= e^{2\pi i \alpha} \chi_{cos}(\tau, \alpha, \beta).
\end{aligned} \tag{3.3}$$

This is a Jacobi form in three variables, of weight zero, and with indices given by the above transformation rules.

To analyze the various parts of the spectrum that contribute to the path integral result, we Poisson resum on the integer m only in equation (3.1) to go to a Hamiltonian picture:

$$\chi_{cos} = \sqrt{k\tau_2} \sum_{n,w} \int \int ds_1 ds_2 \frac{\theta_{11}(\tau, s_2 + s_1\tau - \alpha \frac{k+1}{k} + \beta)}{\theta_{11}(\tau, s_2 + s_1\tau - \frac{\alpha}{k} + \beta)} e^{2\pi i \alpha w} q^{\frac{(n-k(w+s_1))^2}{4k}} \bar{q}^{\frac{(n+k(w+s_1))^2}{4k}} e^{-2\pi i s_2 n}.$$

The details of the intermediate steps follow [10] closely, and we will therefore be brief. We expand the theta-function in denominator and numerator, relabel summation variables, introduce the integral over the radial momentum s , and perform the integration over the holonomies $s_{1,2}$ to find:

$$\begin{aligned}
\chi_{cos} &= \frac{1}{\pi} \frac{1}{\eta^3} \sum_{m,v,w} \int_{-\infty-i\epsilon}^{+\infty-i\epsilon} \frac{ds}{2is+v} (q^{is+\frac{v}{2}} \bar{q}^{is+\frac{v}{2}} - 1) \\
&(-1)^m q^{\frac{(m-\frac{1}{2})^2}{2}} S_{v+m-kw-1} q^{-vw+kw^2} z^{m-1/2} z^{-\frac{v}{k}+2w} (q\bar{q})^{\frac{s^2}{k}+\frac{v^2}{4k}} y^{v-kw},
\end{aligned} \tag{3.4}$$

where the special function $S_r(q)$ is defined by the formula:

$$S_r(q) = \sum_{n=0}^{+\infty} (-1)^n q^{\frac{n(n+2r+1)}{2}}. \tag{3.5}$$

As in [10], we split this result into a holomorphic piece and a remainder term. The holomorphic piece is equal to:

$$\begin{aligned}
\chi_{cos,hol} &= \frac{1}{\pi} \frac{1}{\eta^3} \sum_{m,v,w} \int_{-\infty-i\epsilon}^{+\infty-i\epsilon} \frac{ds}{2is+v} (1 - S_{v+m-kw-1} + S_{v+m-kw-1} q^{is+\frac{v}{2}} \bar{q}^{is+\frac{v}{2}}) \\
&(-1)^m q^{\frac{(m-\frac{1}{2})^2}{2}} q^{-vw+kw^2} z^{m-1/2} z^{-\frac{v}{k}+2w} y^{v-kw} (q\bar{q})^{\frac{s^2}{k}+\frac{v^2}{4k}},
\end{aligned} \tag{3.6}$$

which we can massage, using the properties $q^r S_r = S_{-r}$ and $S_r + S_{-r-1} = 1$ for the special function S_r , into a contour integral:

$$\begin{aligned}
\chi_{cos,hol} &= \frac{1}{\pi} \frac{1}{\eta^3} \sum_{m,v,w} \left(\int_{-\infty-i\epsilon}^{+\infty-i\epsilon} - \int_{-\infty-i\epsilon+i\frac{k}{2}}^{+\infty-i\epsilon+i\frac{k}{2}} \right) \frac{ds}{2is+v} S_{-v-m+kw} (-1)^m q^{\frac{(m-\frac{1}{2})^2}{2}} q^{-vw+kw^2} z^{m-1/2} z^{-\frac{v}{k}+2w} \\
&y^{v-kw} (q\bar{q})^{\frac{s^2}{k}+\frac{v^2}{4k}}.
\end{aligned} \tag{3.7}$$

The contour integral is easily performed. We pick up poles when the radial momentum is equal to the angular momentum $2is+v=0$ for values $2is$ in the interval 0 to $-(k-1)$. We therefore find the discrete character contributions:

$$\chi_{cos,hol} = \sum_{\gamma=0}^{k-1} \frac{1}{\eta^3} \sum_{m,w} S_{-\gamma-m+kw} (-1)^m q^{\frac{(m-\frac{1}{2})^2}{2}} q^{-\gamma w+kw^2} z^{m-1/2} z^{-\frac{\gamma}{k}+2w} y^{\gamma-kw}, \tag{3.8}$$

which is equal to:

$$\chi_{cos,hol} = \sum_{\gamma \in \{0, \dots, k-1\}} \sum_w \frac{i\theta_{11}(\tau, \alpha)}{\eta^3} \frac{q^{kw^2} q^{-w\gamma} z^{2w - \frac{\gamma}{k}}}{1 - zq^{kw-\gamma}} y^{\gamma-kw} \quad (3.9)$$

$$= \frac{1}{k} \sum_{\gamma, \delta \in \mathbb{Z}_k} e^{\frac{2\pi i \gamma \delta}{k}} \frac{i\theta_{11}(\tau, \alpha)}{\eta^3} \sum_{w \in \mathbb{Z}} \frac{q^{\frac{(kw+\gamma)^2}{k}} z^{2\frac{kw+\gamma}{k}}}{1 - z^{\frac{1}{k}} q^{w+\frac{\gamma}{k}} e^{\frac{2\pi i \delta}{k}}} y^{-(\gamma+kw)}, \quad (3.10)$$

where we made the periodicity in the variable γ manifest in the last line. The non-holomorphic remainder term can be rewritten as:

$$\chi_{cos,rem} = -\frac{1}{\pi\eta^3} \sum_{m,n,w} \int_{-\infty-i\epsilon}^{+\infty-i\epsilon} \frac{(-1)^m ds}{2is + n + kw} q^{\frac{(m-1/2)^2}{2}} z^{m-\frac{1}{2}} y^n q^{\frac{s^2}{k} + \frac{(n-kw)^2}{4k}} z^{\frac{kw-n}{k}} \bar{q}^{\frac{s^2}{k} + \frac{(n+kw)^2}{4k}}. \quad (3.11)$$

We see that asymptotically, the global $U(1)$ symmetry has the interpretation of measuring angular momentum on the cigar coset.

It can straightforwardly be checked that the full path integral result for the twisted axial coset elliptic genus can be written in terms of generalized Appell functions as follows:

$$\begin{aligned} \chi_{cos}(q, z, y) &= \frac{1}{k} \frac{i\theta_{11}(\tau, \alpha)}{\eta^3} \sum_{\gamma, \delta \in \mathbb{Z}_k} e^{\frac{2\pi i \gamma \delta}{k}} q^{\frac{\gamma^2}{k}} z^{\frac{2\gamma}{k}} y^{-\gamma} z^{-1} q^{-\gamma} \hat{A}_{2k}(z^{\frac{1}{k}} q^{\frac{\gamma}{k}} e^{\frac{2\pi i \delta}{k}}, q^{-k+2\gamma} z^2 y^{-k}; q) \\ &= \frac{1}{k} \frac{i\theta_{11}(\tau, \alpha)}{\eta^3} \sum_{\gamma, \delta \in \mathbb{Z}_k} e^{-\frac{2\pi i \gamma \delta}{k}} q^{-\frac{\gamma^2}{k}} y^{-\gamma} \hat{A}_{2k}(z^{\frac{1}{k}} q^{\frac{\gamma}{k}} e^{\frac{2\pi i \delta}{k}}, z^2 y^{-k}; q). \end{aligned} \quad (3.12)$$

These generalized Appell functions were defined and analyzed in [8]. It was rigorously proven there that they are real Jacobi forms in three variables [8]. Dressed with the theta-functions, eta-functions, and the prefactors, the modular transformation properties of the generalized Appell functions match those of our path integral result for the twisted elliptic genus.

4 Orbifold, spectrum, and spectral asymmetry

In the previous section, we identified discrete and continuous character contributions to the path integral result, and matched both of these onto the theory of mock modular forms and their modular completion. In this section, we would like to look at the physical interpretations of these expressions in a bit more detail. We will relate the model to the one discussed in [10] and generalize the latter to include the global $U(1)$ twist. We give an interpretation of the holomorphic part in terms of individual free field contributions, and in terms of characters. We also remark on the global $U(1)$ charge as well as on how to derive the spectral density of the non-holomorphic contributions via an independent method.

4.1 Relation to its \mathbb{Z}_k orbifold

We wish to relate the previous result to the one obtained in [10]. In order to do so, we can start with the result we have above, and perform a \mathbb{Z}_k orbifold, where \mathbb{Z}_k is a subgroup of the $U(1)_R$ symmetry. We perform this orbifold as in [18], but the extra β -dependence in the ellipticity properties of our twisted elliptic genus leads to an extra y -dependence in the orbifold formula. Another way to understand this dependence is by realizing that we introduce twisted sectors by spectral flow. Spectral flow changes the boundary conditions on the supercurrents, and therefore on the fermions. Since the fermions contribute to the global $U(1)$ charge, twisting them also gives rise to an extra y -dependence in the phase. Taking this into account, we obtain the

expression:

$$\begin{aligned}
\chi_{orb,hol} &= \frac{1}{k} \sum_{\tilde{\gamma}, \tilde{\delta} \in \mathbb{Z}_k} (-1)^{\tilde{\gamma}+\tilde{\delta}} e^{2\pi i \frac{\tilde{\gamma}\tilde{\delta}}{k}} q^{\frac{\tilde{\gamma}^2}{2} + \frac{\tilde{\delta}^2}{k}} z^{\tilde{\gamma} + \frac{2\tilde{\gamma}}{k}} y^{-\tilde{\gamma}} \frac{i\theta_{11}(\tau, \alpha + \tilde{\gamma}\tau + \tilde{\delta})}{\eta^3} \\
&\quad \sum_{\gamma \in \{0, \dots, k-1\}, m \in \mathbb{Z}} \frac{q^{km^2} q^{-m\gamma} z^{2m - \frac{\gamma}{k}} e^{-2\pi i \tilde{\delta} \frac{\gamma}{k}} q^{\tilde{\gamma}(2m - \frac{\gamma}{k})}}{1 - zq^{km - \gamma + \tilde{\gamma}}} y^{\gamma - km} \\
&= \sum_{\gamma=0}^{k-1} \frac{i\theta_{11}(\tau, \alpha)}{\eta^3} \sum_{m \in \mathbb{Z}} \frac{q^{km^2} q^{+m\gamma} z^{2m + \frac{\gamma}{k}}}{1 - zq^{km}} y^{-km} \\
&= \frac{i\theta_{11}(\tau, \alpha)}{\eta^3} \sum_{m \in \mathbb{Z}} \frac{q^{km^2} z^{2m} y^{-km}}{1 - z^{\frac{1}{k}} q^m}. \tag{4.1}
\end{aligned}$$

This is the coset conformal field theory \mathbb{Z}_k orbifold whose path integral was computed in [10]. Here, we have added a chemical potential coupling to an extra global $U(1)$ symmetry. The non-holomorphic remainder term can also be computed by orbifolding, or as in [10] directly from the path integral result. We find:

$$\chi_{orb,rem} = -\frac{1}{\pi} \frac{1}{\eta^3} \sum_{m \in \mathbb{Z}} (-1)^m q^{\frac{(m - \frac{1}{2})^2}{2}} z^{m-1/2} \sum_{w \in \mathbb{Z}} \sum_{v \in \mathbb{Z}} y^{kw} z^{\frac{v}{k} - 2w} q^{kw^2 - vw} \int_{-\infty - i\epsilon}^{+\infty - i\epsilon} \frac{ds}{2is + v} (q\bar{q})^{\frac{s^2}{k} + \frac{v^2}{4k}}.$$

Asymptotically, the global $U(1)$ charge corresponds to winding number. The complete path integral result is:

$$\chi_{orb} = \sum_{m,w} \int_0^1 ds_1 ds_2 \frac{\theta_{11}(\tau, s_1\tau + s_2 - \frac{k+1}{k}\alpha + \beta)}{\theta_{11}(\tau, s_1\tau + s_2 - \frac{1}{k}\alpha + \beta)} e^{\frac{2\pi i \alpha w}{k}} e^{-\frac{\pi}{k\tau_2} |(m+ks_2) + \tau(w+ks_1)|^2}, \tag{4.2}$$

$$= \frac{i\theta_{11}(\tau, \alpha)}{\eta^3} \hat{A}_{2k}(z^{\frac{1}{k}}, z^2 y^{-k}; q), \tag{4.3}$$

which is the result of [10], dressed with a twist.

If we were to apply the \mathbb{Z}_k orbifold procedure once more, we would recuperate the twisted axial coset partition function. We note that the axial coset result corresponds by T-duality [21][22] to $N = 2$ Liouville theory at radius $R = \sqrt{\alpha'/k}$ and exhibits a single ground state which is in accord with the Witten index calculation of [23], which states that the number of ground states is equal to the radius divided by $\sqrt{\alpha'/k}$. The orbifolded coset has k ground states as can be seen by putting the twists to zero, $\alpha = 0 = \beta$. There are as many ground states as in $N = 2$ Liouville theory at radius $R = \sqrt{\alpha'/k}$ [23]³.

4.2 Interpretation

In this subsection we comment on the contribution of individual states and discuss the $N = 2$ superconformal character content of the holomorphic part of the elliptic genus.

Regularized individual contributions

We can compare the interpretation of the elliptic genus computed here to that given in [10]. Indeed, the free field interpretations of the orbifold result obtained in [10] have counterparts for the axial coset. For instance, under the assumptions that $|q| < |z| < 1$ and $y = 1$, we can expand the holomorphic part of the axial coset as follows (starting from equation (3.10))

$$\chi_{orb,hol} = \frac{1}{k} \sum_{\gamma, \delta \in \mathbb{Z}_k} e^{2\pi i \frac{\gamma\delta}{k}} \frac{i\theta_{11}(\tau, \alpha)}{\eta^3} \left(\sum_{km+\gamma \geq 0, p \geq 0} - \sum_{km+\gamma < 0, p < 0} \right) q^{\frac{(km+\gamma)^2}{k}} z^{2\frac{km+\gamma}{k}} z^{\frac{p}{k}} e^{2\pi i \frac{p\delta}{k}} q^{\frac{p(km+\gamma)}{k}}.$$

³Reference [23] regularizes $N = 2$ Liouville theory such that there are k ground states. One of them belongs to the family of delta-function normalizable states. See e.g. [24] for a discussion. In our context, the delta-function normalizable ground state leads to a minor ambiguity in how to split the elliptic genus into a holomorphic part and a non-holomorphic remainder. This ambiguity is of little consequence.

We can put $km + \gamma = -w$ and $p = w + kn$ to find:

$$\begin{aligned}\chi_{orb,hol} &= \frac{1}{k} \sum_{\gamma, \delta \in \mathbb{Z}_k} e^{2\pi i \frac{\gamma \delta}{k}} \frac{i\theta_{11}(\tau, \alpha)}{\eta^3} \left(\sum_{w \leq 0, w+kn \geq 0} - \sum_{w > 0, w+kn < 0} \right) z^{\frac{(-w+kn)}{k}} e^{2\pi i \frac{w \delta}{k}} q^{-wn} \\ &= \frac{i\theta_{11}(\tau, \alpha)}{\eta^3} \left(\sum_{w \leq 0, w+kn \geq 0} - \sum_{w > 0, w+kn < 0} \right) z^{\frac{kn-w}{k}} q^{-nw}\end{aligned}\quad (4.4)$$

Clearly, this is very much like equation (14) in [10], with the important difference that we are at the inverse radius. As a consequence, we are summing individual contributions over different wedges of the (momentum, winding) plane.

Discrete character sum

The interpretation of the holomorphic part (at $y = 1$) as a sum over discrete characters on the $N = 2$ superconformal algebra is as follows. We consider Ramond ground states of spin j and R-charge $\frac{2j-1}{k} - \frac{1}{2}$ with $N = 2$ superconformal character (see e.g. [22]):

$$ch_d^{\tilde{R}}(j; \tau, \alpha) = z^{\frac{2j-1}{k}} \frac{1}{1-z} \frac{i\theta_{11}(\tau, \alpha)}{\eta^3}, \quad (4.5)$$

and spectrally flow them $-(2j-1)$ units to obtain:

$$ch(j; \tau, \alpha) = z^{\frac{1-2j}{k}} \frac{1}{1-zq^{1-2j}} \frac{i\theta_{11}(\tau, \alpha)}{\eta^3}. \quad (4.6)$$

We recognize this as the $m = 0$ contribution to the holomorphic part of the axial coset elliptic genus in equation (3.9) when summed over the spins $2j-1 = 0, \dots, k-1$. The full holomorphic contribution is obtained by summing over the extension of these characters by spectral flow by multiples of the level k . That settles the $N = 2$ superconformal content of the holomorphic contribution.

4.3 Spectral asymmetry

Finally, we provide an independent way to derive the measure of the non-holomorphic contribution to the elliptic genus. The origin of the remainder contribution is a mismatch in the spectral density of right-moving bosons and right-moving fermions. If we concentrate on right-movers only, the elliptic genus reduces to a Witten index. The fact that we can obtain a contribution to the elliptic genus from a continuum of modes due to a mismatch in the spectral density of boson and fermions is known.

The relative spectral density between the two right-moving Ramond sectors can be read off from the ratio of reflection amplitudes in these sectors. The reflection amplitudes are given by [22]:

$$R^\pm(j, m\bar{m}) = \frac{\Gamma(-2j+1)\Gamma(1+\frac{2j-1}{k})\Gamma(j+m \mp 1/2)\Gamma(j-\bar{m} \pm 1/2)}{\Gamma(2j-1)\Gamma(1-\frac{2j-1}{k})\Gamma(-j+1+m \mp 1/2)\Gamma(-j+1-\bar{m} \pm 1/2)}, \quad (4.7)$$

where for continuous modes we put $j = \frac{1}{2} + is$ where $s \in \mathbb{R}$ and where m, \bar{m} are the left- and right-moving momentum respectively. The spectral asymmetry (or difference in spectral densities) in the two right-moving Ramond sectors is then given by:

$$\Delta\rho(s) = \rho_+(s) - \rho_-(s) = \frac{1}{2\pi i} \frac{d}{ds} \log \frac{R^+}{R^-}. \quad (4.8)$$

Using these formulae, and the fact that we change Ramond sectors for the right-movers only, we obtain a spectral asymmetry:

$$\Delta\rho(s) = \frac{1}{2\pi} \left(\frac{1}{is - \bar{m}} - \frac{1}{is + \bar{m}} \right). \quad (4.9)$$

This is a spectral measure on the half-line $s \in [0, \infty[$. If we integrate the measure against an even function of the radial momentum s , the measure on the full line becomes

$$-\frac{1}{2\pi} \frac{1}{is + \bar{m}} \quad (4.10)$$

which agrees with the measure in the non-holomorphic remainder function, including the normalization (after the appropriate identification $v = 2\bar{m}$). Thus, that provides a direct justification of the measure in the non-holomorphic contribution to the elliptic genus. It gives a direct physical interpretation to the remainder function of [8].

5 Conclusions

In this paper we have given a detailed path integral derivation of the elliptic genus of a non-compact conformal field theory, further twisted by a global $U(1)$ symmetry. We identified the short, discrete contributions with a mock modular form, and the long, continuous contributions as arising from a difference in spectral densities for right-moving fermions and bosons. The whole result is a (non-holomorphic) Jacobi form in three variables. It is possible to generate many further examples of (twisted) elliptic genera, and mock modular forms related to Jacobi forms by orbifolding combinations of non-compact and compact $N = 2$ superconformal models. It will be interesting to further investigate this class of forms. In particular, they will have applications to checks on mirror symmetry for non-compact Gepner and Landau-Ginzburg models (see e.g. [25][24]), including the long multiplet sector. It will also be interesting to attempt to apply these ideas to prove duality properties of black hole entropy counting functions from first principles.

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